Combinatorics 3

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Problem 2.4 Compute the cardinality of the following sets.

(i) Two digit positive odd integers.
(ii) Subsets of S = { a, b, c, d } with cardinality 2.
(iii) Prime numbers whose base-ten digits sum to ten.
Be careful, some have three digits

(i) __ __ → 1, 3, 5, 7, 9

9 5

Total number of pos. odd integers = 45

(ii) $|S| = 4 \to |B| = 2$

$6 = {}^4C_2 \leftarrow \{a,b\}, \{a,c\}, \{a,d\},$
$\{ \ldots\ldots$

(iii) __ __

9 1 ✗
7 3 ✓
5 5 ✗
3 7 ✓
1 9 ✓

x y z

1

$x+y=9$

1 8 1
2 7 1
3 6 1
4 5 1
5 4 1
6 3 1
7 2 1
8 1 1
9 0 1

Prove that

$$A \Delta (B \Delta C) = (A \Delta B) \Delta C$$

Ans:- $A \Delta ((B-C) \cup (C-B))$

$= A \Delta ((B \cap C^c) \cup (C \cap B^c))$

$= (A \cap ((B \cap C^c) \cup (C \cup B^c))^c)$
$\cup (A^c \cap ((B \cap C^c) \cup (C \cap B^c)))$

$= (A \cap ((B \cap C^c)^c \cap (C \cap B^c)^c))$
$\cup (A^c \cap ((B \cap C^c) \cup (C \cap B^c)))$

$(A \Delta B) \Delta C$

$= ((A \cap B^c) \cup (A^c \cap B)) \Delta C$

$= C \Delta ((A \cap B^c) \cup (A^c \cap B))$

$= (C \cap ((A \cap B^c)^c \cap (B \cap A^c)^c))$
$\cup (C^c \cap ((A \cap B^c) \cup (B \cap A^c)))$

$(B \cap C^c)^c = B^c \cup C$

$A \cap (B^c \cup C)$

$(B \cap A^c)^c$
$= B^c \cup A$
$C \cap (B^c \cup A)$

Venn Diagram

$S_1 = \{ 2^n : n \in \{1, 2, \ldots, N\} \}$ in terms of N

$S_2 = \{ 2^{2^n} : n \in \{1, 2, \ldots, N\} \}$

$|S_1 \cap S_2|_{max} = ?$ $|S_1 \cap S_2|_{min} = ?$

$|S_1 \cap S_2|_{max} = ?$ $|S_1 \cap S_2|_{min}$?

$$S_1 \cap S_2 = \{ 2^n = 2^{2^k} : n \in \{1, 2, .., N\}, k \in \{1, 2, .., N\} \}$$
$$= \{ n = 2^k : n \in \{1, 2, .. N\}, k \in \{1, .. N\} \}$$

For $N = 1$, k has to be 0 and $n = 1$ not possible

$|S_1 \cap S_2| = 0$, $|S_1 \cap S_2|_{min} = 0$

$n < 2^n$ for $n \geq 1$

for $k < n$ we can have $2^k = n$

$|S_1 \cap S_2|_{max} = \log_2 N$

$N = 4$, $n = 2, k = 1$; $n = 4, k = 2$ $\Rightarrow \log_2 4 = \log_2 N$

$\boxed{a^b = c \quad b = \log_a c}$

Q:- Prime numbers whose base-10 digits sum to 10. $\Rightarrow$ Infinite (later)

Ans:- dig-sum(19) = 1 + 9 = 10

Q:- $A \cup B = A \cap C$ $\xrightarrow{\text{Output}}$ True / $\xrightarrow{\text{Output}}$ False

$\{ x : \underbrace{A \cup B = A \cap C}_{\text{Conditional statement}} \}$ with boolean output

Boolean Operation & Operator:-

Operator is $\bullet$

Let a, b be two variables then, $a \bullet b$ is another variable with a boolean operator and a boolean operation is done among a and b

Like '+' is a boolean operator   a+b is the boolean operation

'−', '÷', '×' " "

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'−', '÷', '×' " "

A^c is not a boolean operation as 'c' is done on one variable
→ This is an unary operation

$F = \{x_1, x_2, x_3, x_4, \ldots\}$ → Fibonacci

$x_1 = 0$, $x_2 = 1$ (generally)

$x_3 = x_2 + x_1$, $x_4 = x_3 + x_2$, $x_n = x_{n-1} + x_{n-2}$

Q:- Suppose we have a set S with $n \geq 0$ elements. Find the formula for the number of different subsets of S that has k elements

Ans:- ^nC_k = as we will be choosing only k elements from n elements (no ordering needed as it is a subset)

$n = 3$, $k = 2$
$\{a, b, c\}$
$\{a, b\}$, $\{b, c\}$, $\{c, a\}$